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SINGULARITIES OF VISCOSITY SOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS

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1. Introduction

In this report we discuss some aspects of qualitative theory for fully nonlinear second order elliptic equations. By qualitative theory we understand the circle of questions concerning removability of singularities of solutions, Liouville-type theorems, characterisation of behaviour of solutions near singularities, potential theory, and so on. Currently rather complete answers to such questions are available for *linear* and *quasilinear* equations of the form

$$\operatorname{div}\mathbf{A}(x,u,Du)+B(x,u,Du)=0.$$

The equation is (degenerately) elliptic if

$$\big(\mathbf{A}(x,r,\xi),\xi\big)\geq 0 \quad \text{for} \quad \text{all} \quad \xi\in\mathbf{R}^n.$$

The pioneer papers for these classes of equations are due to Serrin and Maz'ya in the 1960-s. For the current state of art we refer to the monographs [23], [12], [19], [20], [21]. The main tools for quasilinear equations are integral estimates for *Sobolev weak* solutions. Sometimes such estimates are very subtle and not easy to prove.

After the fundamental papers by Crandall, Lions, Ishii, Jensen, Caffarelli, and Trudinger, we have the flexible notion of *viscosity* generalised solution for *fully nonlinear* (nonlinear on the second derivatives) elliptic equations. In their papers the basic questions of existence, uniqueness, and regularity for the elliptic equations of the form

$$F(x,u,Du,D^2u)=0$$

have been resolved. Such equation is (degenerately) elliptic if

$$\sum_{i,j=1}^{n}\frac{\partial F}{\partial S_{ij}}(x,r,\eta,S)\xi_i\xi_j\geq 0 \quad \text{for} \quad \text{all} \quad \xi\in\mathbf{R}^n.$$

Here we report on our attempts to develop the qualitative theory for viscosity solutions. The main difficulty is that viscosity solutions do not have an integral nature similar to the distributional or weak solutions. However, it was possible to understand some questions of qualitative theory rather completely. In what follows we describe some results [15], [16], [17], [18] in this direction.

Our main topic will be the singularities of viscosity solutions.This does not exhaust all qualitative theory for PDEs. Recently results on Liouville and Phragmen-Lidelöf type theorems were obtained by I. Capuzzo Dolcetta, A. Cutri, and F. Leoni [11] [5], [6], [10].

This report is based on the talk I gave in October 2001 at the conference in RIMS, Kyoto. I wish to thank Hitoshi Ishii and Shigeo Koike for the invitation to attend it. I also wish to thank Shigeo for his kind hospitality during my visit to Kyoto and the University of Saitama. I also thank Hitoshi Ishii for his friendly patience during preparation of this paper.

2. Fully nonlinear equations and viscosity solutions

Let $(\cdot,\cdot)$ be the Euclidean inner product in $\mathbf{R}^n$, $n \geq 2$. $B(x,R)$ denotes an open ball in $\mathbf{R}^n$ with centre x and radius R, $B_R = B(0,R)$. By $\mathbf{S}^n$, $n \geq 2$, we denote the space of real $n \times n$ symmetric matrices equipped with its usual order; that is for $N \in \mathbf{S}^n$ the condition

$$N \geq 0$$

means that

$$(Nx, x) \geq 0 \quad \text{for} \quad \text{all} \quad x \in \mathbf{R}^n.$$

In the equation

$$F(D^2u) = 0$$

we have $F : \mathbf{S}^n \to \mathbf{R}^1$. We will assume that F is a *uniformly elliptic operator.* That is, there are two constants

$$A \geq a > 0$$

(which are called the *ellipticity constants*), such that for any $M \in \mathbf{S}^n$

$$a\,\mathrm{trace}(N) \leq F(M+N) - F(M) \leq A\,\mathrm{trace}(N) \quad \text{for} \quad \text{all} \quad N \geq 0,$$

or equivalently

$$a\,id \leq \left[\frac{\partial F(M)}{\partial M_{ij}}\right] \leq A\,id.$$

The ratio λ,

$$\lambda = \frac{A}{a}, \quad \lambda \geq 1,$$

is called the *ellipticity of F*. Examples of fully nonlinear uniformly elliptic equations arising in applications are the Bellman and Isaacs equations.

Important operators for the viscosity theory (and for our work) are the *Pucci extremal operators* $\mathcal{P}^{\pm}_{\lambda}$. If μ_j, $j = 1, \ldots, n$ are the eigenvalues of

$M \in \mathbf{S}^n$ and $\lambda \geq 1$, then

$$\mathcal{P}^+_\lambda(M) = \sup_{id \leq A \leq \lambda id} \left(\sum_{i,j=1}^n A_{ij} M_{ij} \right) = \lambda \sum_{\mu_j \geq 0} \mu_j + \sum_{\mu_j < \leq 0} \mu_j,$$

$$\mathcal{P}^-_\lambda(M) = \inf_{id \leq A \leq \lambda id} \left(\sum_{i,j=1}^n A_{ij} M_{ij} \right) = \sum_{\mu_j \geq 0} \mu_j + \lambda \sum_{\mu_j \leq 0} \mu_j.$$

For an arbitrary uniformly elliptic operator F with the ellipticity λ, the following property of viscosity sub- and supersolutions is well known:

$$F(D^2u) \geq 0 \Longrightarrow \mathcal{P}^+_\lambda(D^2u) \geq -F(0),$$
$$F(D^2u) \leq 0 \Longrightarrow \mathcal{P}^-_\lambda(D^2u) \leq -F(0).$$

The *fundamental solutions* E^+, e^+ to the operator $\mathcal{P}^+_\lambda$ are defined by

$$E^+(x) = E^+_\lambda(x) = \begin{cases} \dfrac{1}{|x|^{\frac{(n-1)}{\lambda}-1}} & \text{if } \ 1 \leq \lambda < n-1 \\ -\log|x| & \text{if } \ \lambda = n-1 \\ -|x|^{1-\frac{(n-1)}{\lambda}} & \text{if } \ n-1 < \lambda, \end{cases}$$

$$e^+(x) = e^+_\lambda(x) = \begin{cases} \dfrac{-1}{|x|^{\lambda(n-1)-1}} & \text{if } \ \lambda \geq 1 \text{ and } n \geq 3 \\ \dfrac{-1}{|x|^{\lambda-1}} & \text{if } \ \lambda > 1 \text{ and } n = 2 \\ \log|x| & \text{if } \ \lambda = 1 \text{ and } n = 2. \end{cases}$$

Note that

$$E^+_\lambda \neq -e^+_\lambda \quad \text{if} \quad \lambda > 1.$$

Using the rotational invariance of the Pucci extremal operators, it is easy to check that E^+, e^+ satisfy

(2.1) $$\mathcal{P}^+_\lambda(D^2u) = 0 \quad \text{in} \quad \mathbf{R}^n \setminus \{0\}.$$

It is only (2.1) that justifies the term "fundamental solution". As a direct consequence of the comparison principle in spherical shells any radial solution to (2.1) has either the form $cE^+ + d$, or $ce^+ + d$, where $c \geq 0$, $d \in \mathbf{R}^1$. We define the fundamental solutions E^-, e^-, to the operator $\mathcal{P}^-_\lambda$ by

$$E^- = -E^+, \quad e^- = -e^+.$$

We will consider only the operator $\mathcal{P}^+_\lambda$ Using the equality

$$\mathcal{P}^+_\lambda(M) = -\mathcal{P}^-_\lambda(-M)$$

it is easy to formulate and prove results for $\mathcal{P}^-_\lambda$ parallel to the results for $\mathcal{P}^+_\lambda$.

In several our theorems below we impose the condition

$$\lambda \leq n-1$$

for the operators $\mathcal{P}_\lambda^\pm$ (or for $F(D^2u)$). It is completely analogous to the condition

$$p \leq n$$

for the p-Laplacian

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2}Du\right), \quad p > 1,$$

or to the well known growth restriction for general quasilinear operators in divergence form. For

$$\lambda > n - 1, \tag{2.2}$$

the fundamental solution E_λ^+ for the operator $\mathcal{P}_\lambda^+$ is Hölder continuous at the nonremovable singularity, as in the case for the fundamental solution for the p-Laplacian for $p > n$. Consequently, in removability statements like, say, our Theorem 4.1 the absolute value restrictions are no longer sufficient in the case (2.2). Nevertheless, the ideas behind Theorems 4.1-4.3 work for any λ. For example, using "tilting" arguments as in the proof of Theorem 4.1, cf. [15], it is easy to show that if the ellipticity of F satisfies (2.2) and if

$$\left|u(x) - u(0)\right| \leq C\,|x|^\beta$$

for some

$$\beta > 1 - \left((n-1)/\lambda\right)$$

then 0 is a removable singularity for

$$F(D^2u) = 0.$$

The example of $\mathcal{P}_\lambda^+$ and E^+ shows that this condition on β is sharp. Moreover, the proof of the characterisation in Theorem 4.3 can be easily adapted to embrace the case (2.2), see [14] for the case of the p-Laplacian with $p > n$.

3. Singular sets and capacities

For a domain $\Omega \subset \mathbf{R}^n$ and $\lambda \geq 1$, let $\Psi_\lambda(\Omega)$ be the set of all lower semicontinous viscosity solutions to inequality

$$\mathcal{P}_\lambda^+(D^2u) \leq 0 \quad \text{in} \quad \Omega,$$

such that

$$u(x) \not\equiv 0, \quad x \in \Omega.$$

Elements of $\Psi_\lambda(\Omega)$ are called λ-*superharmonic* functions. For example, if for $1 \leq \lambda \leq n-1$ we define by continuity

$$E_\lambda^+(0) = +\infty,$$

then E_λ^+ becomes lower semicontinuous in $\mathbf{R}^n$ and consequently $E_\lambda^+ \in \Psi_\lambda(\mathbf{R}^n)$.

Now we give an equivalent definition in the spirit of potential theory.

Proposition 3.1. *A lower semicontinuous function $u\colon \Omega \to \mathbf{R}^1 \cup \{+\infty\}$, $u(x) \not\equiv +\infty$, is λ-superharmonic if and only if for any subdomain $\Omega' \subset\subset \Omega$ the implication*

$$\{\mathcal{P}_\lambda^+(D^2h) = 0 \text{ in } \Omega', \quad h \le u \text{ on } \partial\Omega'\} \Longrightarrow h \le u \text{ in } \Omega'. \tag{3.1}$$

holds for any such h.

Any function h in (3.1) is $C^{2,\alpha}$-smooth by the Evans-Krylov regularity. The condition

$$h \le u \quad \text{on} \quad \partial\Omega'$$

in (3.1) means that

$$\limsup_{x\to\partial\Omega'} \Big(h(x) - u(x)\Big) \le 0.$$

Proposition 3.1 for $\lambda = 1$ is proved in [13]. In general case it is possible to follow the same lines, cf. also [4].

From Proposition 3.1 the cone $\Psi_1(\Omega)$ is in fact the cone of classical superharmonic functions. The classical (1-) superharmonic functions are essentially in one-to-one correspondence with with the distributions $U \in \mathcal{D}'(\Omega)$ satisfying

$$-\Delta U \ge 0.$$

Now we are going to give a similar characterisation for λ-superharmonic functions when $\lambda > 1$, cf. Proposition 3.2 below. We remind that for a distribution $f \in \mathcal{D}'(\Omega)$, the condition

$$f \ge 0$$

means that

$$\langle f, \eta\rangle \ge 0 \quad \text{for} \quad \text{every} \quad \eta \in C_0^\infty(\Omega) \quad \text{such} \quad \text{that} \quad \eta \ge 0.$$

Every nonnegative distribution is a Radon measure, see e.g. [24].

The set $\Psi_\lambda(\Omega)$ is a convex functional cone, and

$$\Psi_\lambda(\Omega) \subset \Psi_\nu(\Omega), \quad \text{when} \quad \lambda \ge \nu \ge 1.$$

Hence λ-superharmonic functions are harmonic in the classical sense, and in particular

$$\Psi_\lambda(\Omega) \subset L^1_{\text{loc}}(\Omega).$$

Also the viscosity definitions (or characterisation (3.1)) imply that

$$u, v \in \Psi_\lambda(\Omega) \Longrightarrow \min\{u, v\} \in \Psi_\lambda(\Omega).$$

Proposition 3.2. *If $u \in \Psi_\lambda(\Omega)$ then*

$$-\sum_{i,j=1}^n A_{ij} D_{ij} u \ge 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \tag{3.2}$$

for all $A \in G_\lambda$. Conversely, if $U \in \mathcal{D}'(\Omega)$ satisfies (3.2) for all matrices $A \in G_\lambda$ then U is equivalent to a unique $u \in \Psi_\lambda(\Omega)$.

Proposition 3.2 for smooth u follows directly from the definitions via the simple linear algebra. The proof in the general case is based on the suitable smooth approximation and can be found in [15].

As a consequence of Proposition 3.2 we will now prove that for any $u \in \Psi_\lambda(\Omega)$, $\lambda > 1$, all the second derivatives $D_{ij}u$, i, $j = 1, \ldots, n$, are signed Radon measures in Ω, cf. Corollary 3.3. Of course this is not true for classical (1-)superharmonic functions, for which only the combination

$$-\Delta u = -D_{11}u - \cdots - D_{nn}u$$

is a Radon measure. Properties of functions whose Hessian matrices are signed Radon measures have been investigated in the literature, see e.g. [1] and references therein. Thus Corollary 4.3 implies that the results of [1] hold for functions in $\Psi_\lambda(\Omega)$ with $\lambda > 1$.

Corollary 3.3. *If $u \in \Psi_\lambda(\Omega)$, $\lambda > 1$, then the distributional derivatives $D_{ij}u$ are signed Radon measures for all $i, j = 1, \ldots, n$.*

When investigating the local properties of λ-superharmonic functions, we can restrict ourselves to the case $\Omega = B_R$, for some fixed $R > 0$. In what follows we set

$$\Psi_\lambda = \Psi_\lambda(B_R).$$

A set $E \subset\subset B_{R/2}$ is called *λ-polar* if there exists a function $u \in \Psi_\lambda$ such that

$$u|_E = +\infty.$$

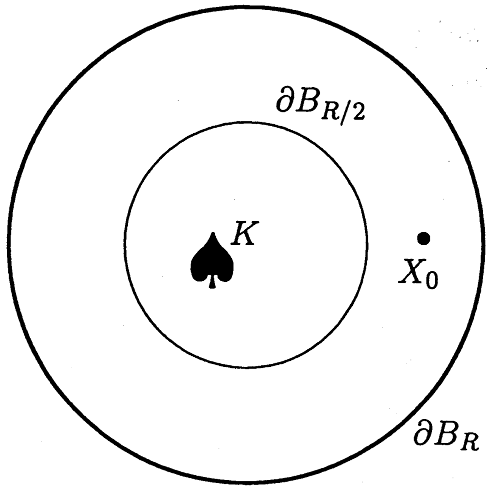
A compact set $K \subset B_{R/2}$ is called *removable for an operator F* if the following implication holds:

$$F(D^2u) = 0 \quad \text{in} \quad B_R \setminus K, \quad u \in L^\infty(B_R) \Longrightarrow F(D^2u) = 0 \quad \text{in} \quad B_R.$$

To study removable sets for fully nonlinear uniformly elliptic operators F, we introduce the capacity suitable for λ-superharmonic functions [17]. The capacity will be defined on sets $E \subset\subset B_{R/2}$. Fix a point X_0,

$$X_0 \in B_R \setminus B_{R/2}, \quad |X_0| = 2R/3.$$

Let $K \subset B_{R/2}$ be a compact set:



First we define the *capacitary potential of* K. Set

$$\mathcal{U}_\lambda(K) = \{u \in \Psi_\lambda : \ u \geq 0 \text{ in } B_R, \ u \geq 1 \text{ on } K\}$$

Define the function $u_K \colon B_R \to \mathbf{R}$ by writing

$$u_K(x) = u_{K,\lambda}(x) = \inf\{u(x) : \ u \in \mathcal{U}_\lambda(K)\}. \tag{3.3}$$

Clearly

$$u_K = 1 \text{ on } K.$$

The capacitary potential $\overline{u}_K$ of K is the lower semicontinuous regularisation of u_K:

$$\overline{u}_K(x) = \liminf_{y \to x} u_K(y), \quad x \in B_R.$$

Standard arguments in viscosity theory [3], [7], [8] give

$$\overline{u}_K \in \Psi_\lambda.$$

Applying the strong minimum principle for the classical superharmonic functions we see that

$$\text{either} \quad \overline{u}_K > 0 \quad \text{in} \quad B_R, \quad \text{or} \quad \overline{u}_K \equiv 0 \quad \text{in} \quad B_R.$$

Moreover, $\overline{u}_K$ is the (upper) Perron solution to the Dirichlet problem

$$\begin{cases} \mathcal{P}_\lambda^+(D^2u) = 0 & \text{in} \quad B_R \setminus K \\ u = 0 & \text{on} \quad \partial B_R \\ u = 1 & \text{on} \quad K. \end{cases} \tag{3.4}$$

Viscosity theory and Evans-Krylov local regularity give

$$u_K = \overline{u}_K \text{ in } B_R \setminus K, \quad u_K \in C^{2,\alpha}_{\text{loc}}(B_R \setminus K).$$

For general K we can only say that

$$\overline{u}_K \leq u_K \text{ in } B_R.$$

But for sufficiently regular K (say, K satisfying the cone condition) problem (3.4) has the unique classical solution [9]. Therefore

$$u_K = \overline{u}_K \text{ in } B_R, \quad u_K \in C(\overline{B}_R) \cap C^{2,\alpha}_{\text{loc}}(B_R \setminus K)$$

for such regular K.

Next we define the λ-*capacity* of a compact set $K \subset B_{R/2}$ as

$$\mathcal{C}_\lambda(K) = \overline{u}_K(X_0) = u_K(X_0).$$

It has the following properties:

$$\mathcal{C}_\lambda(K_1) \le \mathcal{C}_\lambda(K_2) \text{ for } K_1 \subset K_2 \subset B_{R/2},$$
$$\mathcal{C}_\lambda(K_1 \cup K_2) \le \mathcal{C}_\lambda(K_1) + \mathcal{C}_\lambda(K_2) \text{ for any } K_1, K_2 \subset B_{R/2}.$$

These properties essentially follow directly from (3.3). The next important property of $\mathcal{C}_\lambda$ is slightly less trivial. We claim that for a monotone sequence of compact sets $B_{R/2} \supset K_1 \supset K_2 \supset \cdots$ we have

$$\mathcal{C}_\lambda\Big(\bigcap_{j=1}^{\infty} K_j\Big) = \lim_{j\to\infty} \mathcal{C}_\lambda(K_j). \tag{3.5}$$

The proof is omitted.

So far the capacity has been defined only on compact sets. It is monotone, subadditive, and satisfies (3.5) Let us know briefly describe the axiomatic procedure of its extension to arbitrary sets. First define the *outer capacity* for any *open* set $O \subset\subset B_{R/2}$ as

$$\mathcal{C}^*_\lambda(O) = \sup\{\mathcal{C}_\lambda(K) : \ K \subset O, \ K \text{ compact}\}.$$

Then for *arbitrary* $E \subset\subset B_{R/2}$ we set

$$\mathcal{C}^*_\lambda(E) = \inf\{\mathcal{C}^*_\lambda(O) : \ O \supset E, \ O \text{ open}\}.$$

It easy to show that $\mathcal{C}^*_\lambda$ is monotone and subadditive on all subsets of $B_{R/2}$. It is correctly defined on open sets. Moreover, for a compact set $K \subset B_{R/2}$ we have

$$\mathcal{C}^*_\lambda(K) = \mathcal{C}_\lambda(K). \tag{3.6}$$

Next, the abstract arguments allow to derive from the subadditivity, (3.5), and (3.6) that

$$\mathcal{C}^*_\lambda\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \lim_{J\to\infty} \mathcal{C}^*_\lambda\Big(\bigcup_{j=1}^{J} E_j\Big).$$

for any sequence $\{E_j\}$ such that

$$\Big(\bigcup_{j=1}^{\infty} E_j\Big) \subset\subset B_{R/2}.$$

Finally, the Choquet abstract theorem asserts that for any Borell (even more generally, for any Suslin) set $E \subset\subset B_{R/2}$ we have

$$\mathcal{C}^*_\lambda(E) = \sup\{\mathcal{C}_\lambda(K) : \ K \subset E, \ K \text{ compact}\}.$$

Sets with such property are called *capacitable*. In particular, statement (3.6) says that compact sets are capacitable.

In what follows we use the outer capacity $\mathcal{C}^*_\lambda$ for non-compact sets $E \subset\subset B_{R/2}$. However, we omit the star and denote it by $\mathcal{C}_\lambda$.

To illustrate the definitions let us calculate the capacity of the ball B_r, $r < R/2$. Using the radial fundamental solution E^+_λ we derive the formula for the capacitary potential of $\overline{B}_r$, or in other words for the solution to (3.4) with $K = \overline{B}_r$:

$$u_{\overline{B}_r}(x) = \min\left\{1, \frac{E^+_\lambda(x) - E^+_\lambda(R)}{E^+_\lambda(r) - E^+_\lambda(R)}\right\}, \quad \text{for} \quad x \in B_R.$$

Hence from the definition

$$\mathcal{C}_\lambda(B_r) = C(n,\lambda)\frac{r^{\frac{n-1}{\lambda}-1}}{R^{\frac{n-1}{\lambda}-1} - r^{\frac{n-1}{\lambda}-1}} \quad \text{for} \quad 1 \le \lambda < n-1,$$

$$\mathcal{C}_{n-1}(B_r) = C\frac{1}{\log(R/r)} \quad \text{for} \quad \lambda = n-1,$$

$$\mathcal{C}_\lambda(B_r) = C(n,\lambda)\frac{R^{1-\frac{n-1}{\lambda}}}{R^{1-\frac{n-1}{\lambda}} - r^{1-\frac{n-1}{\lambda}}} \quad \text{for} \quad \lambda > n-1.$$

It follows that for $\lambda > n-1$ the singletons have positive capacity. Consequently

$$\mathcal{C}_\lambda(E) = 0 \Longleftrightarrow E = \emptyset.$$

provided $\lambda > n-1$.

Capacities defined for different choices of X_0 are equivalent. Indeed, for any $K \subset B_{R/2}$ the function $\overline{u}_K$ solves uniformly elliptic equation (3.4) in $B_R \setminus B_{R/2}$. Utilising the Krylov-Safonov Harnack inequality we conclude that

$$\frac{1}{C}\,\overline{u}_K(Y_0) \le \overline{u}_K(X_0) \le C\,\overline{u}_K(Y_0) \quad \text{for all} \quad X_0, Y_0 \in B_{R-\delta} \setminus B_{R/2+\delta}$$

for any $\delta > 0$ with a constant $C > 0$, $C = C(n,\lambda,\delta/R)$. For $\lambda = 1$ our capacity $\mathcal{C}_1$ is the classical (electrostatic) capacity for the Laplace operator.

Now we state main theorems on removable sets [17].

Theorem 3.4. *Let $K \subset B_{R/2}$ be a compact set, and let $\lambda \ge 1$. The following statements are equivalent:*
(i) *The set K is λ-polar.*
(ii) *The set K is removable for bounded solutions of the equation*

$$F(D^2u) = 0$$

for all uniformly elliptic operators F with the ellipticity λ.
(iii) $\mathcal{C}_\lambda(K) = 0$.

It is important that Theorem 3.4 allows to obtain some geometric information on removable and polar sets. For this purpose we will need the notions of the Riesz capacities Cap_α and the Hausdorff measures $\mathcal{H}^\alpha$. They can be found e.g. in [22].

Theorem 3.5. *Let $K \subset B_{R/2}$ be a compact set, and let $1 \le \lambda \le n-1$. Then:*
(i) $\mathrm{Cap}_{\frac{n-1}{\lambda}-1}(K) = 0 \Longrightarrow \mathcal{C}_\lambda(K) = 0$.
(ii) $\mathcal{H}^{\frac{n-1}{\lambda}-1}(K) < +\infty \Longrightarrow \mathcal{C}_\lambda(K) = 0$.

4. Isolated singularities

In this section we consider the case when the singular set is an isolated point. The results in this case are quite complete [15], [16]. Let us state some of them.

Theorem 4.1. *Let $u \in C_{\mathrm{loc}}(B_R \setminus \{0\})$ solve*

$$F(D^2u) = 0 \quad in \quad B_R \setminus \{0\}, \tag{4.1}$$

where F is a uniformly elliptic operator with the ellipticity λ, $1 \le \lambda \le n-1$. If

$$u(x) = o\left(E_\lambda^+(x)\right) \quad when \quad x \to 0, \tag{4.2}$$

then the singularity at 0 is removable and u is a solution of (4.1) in the entire ball B_R.

The next result concerns the Pucci operators $\mathcal{P}_\lambda^+$. It sates that any one side bounded solution to the equation

$$\mathcal{P}_\lambda^+(D^2u) = 0$$

in the punctured ball is either extendible to the solution in the entire ball, or can be controlled near the centre of the ball by means of the fundamental solution.

Theorem 4.2. *Let $u \in C^2_{\mathrm{loc}}(B_R \setminus \{0\})$, $u \ge 0$, satisfy*

$$\mathcal{P}_\lambda^+(D^2u) = 0 \quad in \quad B_R \setminus \{0\}, \tag{4.3}$$

where $B_R \subset \mathbf{R}^n$, $n \ge 2$, $1 \le \lambda \le n-1$. Then either the singularity at 0 is removable and u is a classical solution of (4.3) in the entire ball B_R, or there exists a real number $\gamma > 0$ such that

$$u(x) = \gamma E_\lambda^+(x) + O(1), \quad x \to 0,$$

and

$$D^\alpha u(x) = \gamma D^\alpha E_\lambda^+(x) + o\left(\frac{1}{|x|^{\frac{n-1}{\lambda}-1+|\alpha|}}\right), \quad x \to 0,$$

for all multi-indices α with $1 \le |\alpha| \le 2$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

According to the Evans-Krylov estimates, any viscosity solution to (4.3) enjoys $C^{2,\alpha}_{\mathrm{loc}}$ regularity and, consequently, is a classical solution. Because of the lack of differentiability of the matrix function $\mathcal{P}^+_\lambda$ we cannot in general expect the existence of derivatives of order 3 and higher for solutions of (4.3). For $\lambda = 1$ we have

$$\mathcal{P}^+_\lambda(D^2u) = \Delta u.$$

The proof of Theorem 4.2 is based on the scale invariance of the operator and the classical maximum principle. It uses a blow-up construction of Kichenassamy and Véron [14]. Because of the Evans-Krylov regularity estimates it is possible to avoid viscosity solutions entirely in the proof. The condition

$$\lambda \leq n-1$$

has been discussed in section 2.

We conclude with the result on the unconditionally removable isolated singularities [16]. We define

$$q(\lambda) = \frac{n-1+\lambda}{n-1-\lambda}. \tag{4.4}$$

Assume that the function $f : \mathbf{R}^1 \to \mathbf{R}^1$ is continuous and satisfies

$$\begin{aligned} &\limsup_{t\to+\infty} \frac{f(t)}{|t|^{q(\lambda)}} < 0 \\ &\liminf_{t\to-\infty} \frac{f(t)}{|t|^{q(\lambda)}} > 0. \end{aligned} \tag{4.5}$$

Theorem 4.3. *Let F be a uniformly elliptic operator in $\mathbf{S}^n$, $n \geq 3$, with the ellipticity λ, $1 \leq \lambda < n-1$, and let $u \in C_{\mathrm{loc}}(B_R \setminus \{0\})$ be a solution to*

$$F(D^2u) + f(u) = 0 \quad \text{in} \quad B_R \setminus \{0\}, \tag{4.6}$$

where the continuous function f satisfies (4.5). Then u can be defined at 0 as a solution to the equation in (4.6) in the entire ball B_R.

The semilinear case $\lambda = 1$ in Theorem 4.3 was proved by Brezis and Veron in their seminal paper [2]. As a corollary of Theorem 4.3 we obtain that isolated singularities are removable for the fully nonlinear equation

$$\mathcal{P}^+_\lambda(D^2u) - |u|^{q-1}u = 0, \quad q > 1, \tag{4.7}$$

if and only if

$$1 \leq \lambda < n-1 \quad \text{and} \quad q \geq q(\lambda),$$

where $q(\lambda)$ is defined by (4.4). To see that the "only if" part holds it is enough to note the following. For

$$\lambda \geq n-1, \qquad \text{and} \quad \text{any} \quad q > 1,$$

or for

$$1 \leq \lambda < n-1 \quad \text{and} \quad 1 < q < q(\lambda)$$

equation (4.7) has a solution of the form

$$u(x) = \frac{A_1}{|x|^{\frac{2}{q-1}}}, \quad A_1 > 0.$$

For

$$1 < q < \frac{\lambda(n-1)+1}{\lambda(n-1)-1} \quad \text{and} \quad \text{any} \quad \lambda \geq 1$$

equation (4.7) has also a solution of the form

$$u(x) = -\frac{A_2}{|x|^{\frac{2}{q-1}}}, \quad A_2 > 0.$$

Constants $A_{1,2}(\lambda, \Lambda, n, q)$ can easily be calculated.

For further comments on the results similar to Theorems 4.2, 4.3 we refer to [15], [16], [17].

References

[1] P. Aviles, Y. Giga, *Singularities and rank one properties of Hessian measures*, Duke Math. J. **58** (1989), 441–467.

[2] H. Brezis, L. Veron, *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980/81), 1-6

[3] , L. A. Caffarelli, X. Cabre, Fully nonlinear elliptic equations. American Mathematical Society, Providence, RI, 1995.

[4] I. Capuzzo Dolcetta, preprint, 2002.

[5] , I. Capuzzo Dolcetta, A. Cutri, *Hadamard and Lioville theorems*, preprint, 2002.

[6] , I. Capuzzo Dolcetta, A. Cutri, *On the Liouville property for sub-Laplacians*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), 239–256

[7] , M. G. Crandall, H. Ishii, P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.

[8] M. G. Crandall, *Viscosity solutions: a primer*, Viscosity solutions and applications (Montecatini Terme, 1995), 1–43, Lecture Notes in Math., 1660, Springer, Berlin, 1997.

[9] M. G. Crandall, M. Kocan, P. L. Lions, A. Swiech, *Existence results for boundary problems for uniformly elliptic and parabolic fully nonlinear equations*, Electron. J. Differential Equations (1999) **24**.

[10] A. Cutri, *Phragmen-Lindelof theorems*, work in progress.

[11] A. Cutri, F. Leoni, *On the Liouville property for fully nonlinear equations*, Ann. Inst. H. Poincare Anal. Non Lineaire, **17** (2000), 219–245.

[12] J. Heinonen, T. Kilpelainen, O. Martio, Nonlinear potential theory of degenerate elliptic equations. Oxford University Press, New York, 1993.

[13] L. Hörmander, Notions of convexity. Birkhauser Boston, Inc., Boston, MA, 1994.

[14] S. Kichenassamy, L. Veron, *Singular solutions of the p-Laplace equation*, Math. Ann. **275** (1986), 599–615

[15] D. A. Labutin, *Isolated singularities for fully nonlinear elliptic equations*, J. Differential Equations **177** (2001), 49–76.

[16] D. A. Labutin, *Removable singularities for fully nonlinear elliptic equations*, Arch. Ration. Mech. Anal. **155** (2000), 201–214.

[17] D. A. Labutin *Singularities for viscosity solutions of fully nonlinear elliptic equations*, preprint.

[18] D. A. Labutin PhD. Thesis, Australian National University, 2000.

[19] J. Malý, W. P. Ziemer, Fine regularity of solutions of elliptic partial differential equations. American Mathematical Society, Providence, RI, 1997.

[20] , V. A. Kozlov, V. G. Maz'ya, J. Rossmann, Elliptic boundary value problems in domains with point singularities. American Mathematical Society, Providence, RI, 1997.

[21] , V. A. Kozlov, V. G. Maz'ya, J. Rossmann, Spectral problems associated with corner singularities of solutions to elliptic equations. American Mathematical Society, Providence, RI, 2001.
[22] N. S. Landkof, Foundations of modern potential theory. Springer-Verlag, New York-Heidelberg, 1972.
[23] L. Veron, Singularities of solutions of second order quasilinear equations. Longman, Harlow, 1996.
[24] W. P. Ziemer, Weakly differentiable functions. Sobolev spaces and functions of bounded variation. Springer-Verlag, New York, 1989.

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